

# Asymptotic estimate for the number of Gaussian packets on three decorated graphs

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## Abstract

We study a topological space obtained from a graph via replacing vertices with smooth Riemannian manifolds, i.e. a decorated graph. We construct a semiclassical asymptotics of the solutions of Cauchy problem for a time-dependent Schrödinger equation on a decorated graph with a localized initial function. The main term of our asymptotic solution at an arbitrary finite time is the sum of Gaussian packets and generalized Gaussian packets. We study the number of such packets as time goes to infinity. We prove asymptotic estimations for this number for the following decorated graphs: cylinder with a segment, two dimensional torus with a segment, three dimensional torus with a segment. Also we prove general theorem about a manifold with a segment and apply it to the case of a uniformly secure manifold.

## 1 Introduction

Differential equations and differential operators on decorated graphs have been intensively studied over the past thirty years (see, for example, [1], [2], [3], [4] and references therein).

The study of the motion of Gaussian packets arises when considering a Cauchy problem for a time-dependent Schrödinger equation. Let us consider

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the Cauchy problem for a decorated graph, which is a singular space obtained by gluing the ends of segments to the surfaces of dimension two or three. The initial conditions are a Gaussian packet with support on one of the edges. We look for a semiclassical solution (see, for example, [7], [8] and references therein). Upon reaching the end of the segment, the packet forms an expanding wavefront on the surface. If the front reaches another point of gluing then a new Gaussian packet starts to move along the corresponding edge and so on. We are interested in finding the asymptotic behavior of the number of supports of the Gaussian packets on the edges of a decorated graph at time  $T$ . A detailed description of the formulation of this problem is given in [10].

Wavefront propagation on surfaces is associated with the properties of geodesics on the surface. There are two variations. In the first the number of geodesics connecting two points on the surface is finite. As an example we can take a standard sphere with an edge. In this situation we can construct from one-dimensional edges and geodesics on the surfaces an equivalent metric graph and describe the statistics of the distribution of Gaussian packets using our results obtained before (see, e.g., [5], [6]). Theorems on the asymptotic behavior of the number of packets  $N(T)$  and the uniformity of their distribution for this case were proven in [10].

The second variation is more general, where the equivalent geometric graph is infinite. The main question here is how the number of geodesics joining two given points increases as time goes to infinity. The time and the maximum length are synonymous in this context. Some results in this area can be found in [13]. Let  $h$  be a topological entropy for a compact Riemannian manifold  $M$ . R. Mañé has shown in [16] (see references therein) that for quite general situations  $h = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{M \times M} \log CF_T(x, y) d\mu(x) d\mu(y)$ , where  $CF_T(x, y)$  is the number of geodesics joining  $x$  to  $y$  of maximum length  $T$ . But this equation may fail if  $M$  has conjugate points. There are examples where the growth of  $CF_T(x, x)$  is arbitrarily large for some exceptional points  $x \in M$  (see [17]). There are also examples in which the limit in the equation is smaller than the topological entropy for an open set of configurations (see [15]).

A Riemannian manifold is said to be uniformly secure (see [17]) if there is a finite number  $k$  such that all geodesics connecting an arbitrary pair of points in the manifold can be blocked by  $k$  point obstacles. It is proven in [17] that the number of geodesics with length  $\leq T$  between every pair of points

in a uniformly secure manifold grows polynomially as  $T \rightarrow \infty$ . According to the results of M. Gromov and R. Mañé (see [16]), the fundamental group of such a manifold is virtually nilpotent, and the topological entropy of its geodesic flow is zero. Furthermore, if a uniformly secure manifold has no conjugate points, then it is flat. This follows from the virtual nilpotency of its fundamental group either via the theorems of Croke-Schroeder and Burago-Ivanov, or by a more recent work of Lebedeva [18].

In the present article we consider the case where the number of geodesics grows polynomially, for example a compact Riemannian manifold that is uniformly secure. We study in detail two examples: a standard cylinder and a flat torus with one edge. In the situation where the equivalent lengths of the edges of the graph are linearly independent over the field of rational numbers, there is a one-to-one correspondence between the times in which the birth of new Gaussian packets occurs and nonnegative solutions of infinite linear inequality where the right hand side equals  $T$ . Such inequalities arise in number theory, namely in the analysis of the asymptotic behavior of the number of partitions of natural numbers. We can recall the results of G.H. Hardy and S. Ramanujan, J.V. Uspensky, G. Rademacher, P. Erdős and others. In our case, the length of the geodesics will not be a sequence of natural numbers. Therefore, at the first stage of our research we obtain only an upper bound from the classical results. On the other hand, a lower bound was proven in [10], namely for infinite number of edges  $N(T)$  grows faster than any polynomial. Later we found a connection between the problem we had studied and questions arising in the study of Bose-Maslov gas entropy, which in recent years was carried out by V.P. Maslov and V.E. Nazaikinskii. We used a 2013 result of V.E. Nazaikinskii (see [9]) to obtain asymptotic formulas for the logarithm of the number of Gaussian packets on the edge. If linear independence over  $\mathbb{Q}$  does not hold (such situation is certainly possible because we can even have many geodesics with the same length, see e.g. [12]), the resulting asymptotic formula becomes an upper bound for  $N(T)$ . It should be noted that the same results can be obtained using Additive Abstract Prime Number Theorem from [11]. A computer experiment conducted jointly with O.V. Sobolev has confirmed the correctness of the estimates. The simulation results for a decorated graph obtained by attaching a length of the cylinder are also given in [10].

## 1.1 Preliminary remarks and definitions

A *decorated graph* is a topological space, obtained from a metric graph by replacing vertices by smooth manifolds of dimensions two or three. Consider a finite number of smooth complete Riemannian manifolds  $M_j$ , and a number of segments  $\gamma_i$ , endowed with regular parametrization. For each endpoint  $y$  of an arbitrary segment  $\gamma_j$  fix a point  $\tilde{y}$  on one of the manifolds  $M_k$ ; we assume all points  $\tilde{y}$  to be distinct. A decorated graph  $\Gamma_d$  is a quotient space of the disjoint sum  $\bigsqcup_k M_k \bigsqcup_j \gamma_j$  by the equivalence  $y \sim \tilde{y}$ .

The Schrödinger equation on a decorated graph is defined as follows (see [2] and [10] for detailed explanation; the original ideas were presented in [3],[4]).

Let  $V$  be a real valued continuous function on  $\Gamma_d$ , smooth on the edges. Let  $V_j$  and  $V_k$  be restrictions of  $V$  to  $\gamma_j$  and to  $M_k$  respectively. Consider a direct sum  $\widehat{H}_0 = \bigoplus_{j=1}^E \left( -\frac{h^2}{2} \frac{d^2}{dz_j^2} + V_j \right) \bigoplus_{k=1}^V \left( -\frac{h^2}{2} \Delta_k + V_k \right)$  with the domain  $H^2(\Gamma) = \bigoplus_{j=1}^E H^2(\gamma_j) \bigoplus_{k=1}^V H^2(M_k)$ . Here  $\frac{d^2}{dz_j^2}$  is an operator of the second derivative on  $\gamma_j$  with respect to a fixed parametrization with Neumann boundary conditions,  $\Delta_k$  is the Laplace–Beltrami operator on  $M_k$ .

**Definition.** The *Schrödinger operator*  $\widehat{H}$  is a self-adjoint extension of the restriction  $\widehat{H}_0|_L$ , where  $L = \{\psi \in H^2(\Gamma), \quad \psi(y_s) = 0\}$ .

Domain of the operator  $\widehat{H}$  contains functions with singularities in the points  $y_j$ . Namely, let  $G(x, y, \lambda)$  be the Green function on  $M_k$  (integral kernel of the resolvent) of  $\Delta$ , corresponding to the spectral parameter  $\lambda$ . This function has the following asymptotics as  $x \rightarrow y$ :  $G(x, y, \lambda) = F_0(x, y) + F_1$ , where  $F_1$  is a continuous function and  $F_0$  is independent of  $\lambda$  and has the form  $F_0 = -\frac{c_2}{2\pi} \ln \rho$  if  $\dim M = 2$  and  $F_0 = \frac{c_3}{4\pi\rho}$  if  $\dim M = 3$ . Here  $c_j(x, y)$  is continuous,  $c_j(y, y) = 1$ ,  $\rho$  is the distance between  $x$  and  $y$ . The function  $\psi$  from the domain of the operator  $\widehat{H}$  has the following asymptotics as  $x \rightarrow y_j$ :  $\psi = \alpha_j F_0(x) + b_j + o(1)$ .

Now for each endpoint of the segment consider a pair  $\psi(y)$ ,  $h\psi'(y)$  and a vector  $\xi = (u, v)$ ,

$$\begin{aligned} u &= (h\psi'(y_1), \dots, h\psi'(y_{2E}), \alpha_1, \dots, \alpha_{2E}), \\ v &= (\psi(y_1), \dots, \psi(y_{2E}), hb_1, \dots, hb_{2E}). \end{aligned}$$

Consider a standard skew-Hermitian form  $[\xi^1, \xi^2] = \sum_{j=1}^{4E} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2)$  in  $\mathbb{C}^{4E} \oplus \mathbb{C}^{4E}$ . Let us fix the Lagrangian plane  $\Lambda \subset \mathbb{C}^{4E} \oplus \mathbb{C}^{4E}$ . A self-adjoint

extension  $\widehat{H}$  is defined by the coupling conditions  $\xi \in \Lambda$  or equivalently  $-i(I + U)u + (I - U)v = 0$ , where  $U$  is a unitary matrix defining  $\Lambda$  and  $I$  is an identity matrix. We will consider only local coupling conditions, i.e.  $\Lambda = \bigoplus_y \Lambda_y$ , where  $\Lambda_y \subset \mathbb{C}^4$  is defined for each point  $y$  separately.

In Theorem 3.1 of the article [10] it is shown how scattering of a Gaussian packet at the point of gluing of the edge to the surface takes place.

Further analysis of the number of packets on a decorated graph is as follows. Since wavefront propagation occurs along the geodesic, we can construct a matching metric graph for a given decorated graph. Vertices for this new graph are the points of gluing and edges are all geodesics on the surfaces connecting points of gluing and the old edges. We are interested in the number of old Gaussian packets, i.e. those whose support will lie on the edges of the original graph.

The number of packets could change only in those moments of time that have the form of linear combinations of edge travel time. So the number of packets is equal to the number of sets  $\{n_j\}$  satisfying some inequations of this kind:

$$n_1 t_{l_1} + \dots + n_m t_{l_m} \leq T,$$

where  $t_j$  is a travel time of the  $j$ -th edge of equivalent graph and all  $t_j$  are linearly independent over  $\mathbb{Q}$ . In the case where an equivalent graph contains an infinite number of edges we obtain an infinite inequality.

In the work [9] of V. E. Nazaikinskii the formula for  $\ln N(E)$  i.e. entropy of a gas with full energy less than  $E$  was obtained.  $N(E)$  is defined via the number of solutions of an infinite linear inequality, where  $E$  is on the right hand side of the inequality.

Let us state this useful theorem.

**Theorem 1.1** *Let  $N(T)$  be the number of non-negative integer solutions of inequality  $\sum_{i=1}^{\infty} \lambda_i N_i \leq T$  and for sequence  $\lambda_j$  a counting function  $\rho(\lambda) = \#\{j | \lambda_j \leq \lambda\}$  has asymptotics*

$$\rho(\lambda) = c_0 \lambda^{1+\gamma} (1 + O(\lambda^{-\varepsilon})), \varepsilon > 0.$$

*Then*

$$\ln N(T) = (\gamma + 2) \left( \frac{c_0 \Gamma(\gamma + 2) \zeta(\gamma + 2)}{(\gamma + 1)^{\gamma+1}} \right)^{\frac{1}{\gamma+2}} T^{\frac{\gamma+1}{\gamma+2}} (1 + o(1))$$

*as  $T$  goes to infinity.*

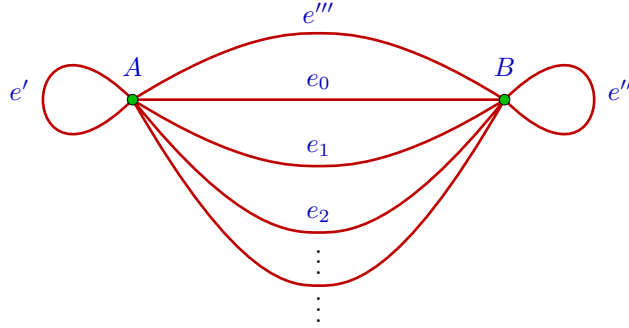
In next two sections we will consider two examples which show how to construct the asymptotics for  $N(T)$  for a given decorated graph.

## 2 Decorated graph, obtained by gluing a segment to a cylinder

Let us consider a circular cylinder with a length of a circle equaling  $b$ . Points  $A$  and  $B$  lie on a ruling of the cylinder at the distance  $a$  from each other. The wave front that begins to spread from the point  $A$  will reach the point  $A$  again in the time of the form  $nb$ . Point of the front which is the first to return to the point  $A$  will pass the cycle  $e'$  with the passage time  $t' = b$ . Similarly, we define a cycle  $e''$  from  $B$  to  $B$  with the passage time  $t'' = b$ .

Let us assume that wavefront propagates from the point  $A$  reaches the point  $B$  at time  $\sqrt{(kb)^2 + a^2}$  ( $k \geq 0$ ). One of points on wave front that reaches  $B$  will pass way  $e_k$  during the time  $t_k = \sqrt{(kb)^2 + a^2}$ . Also, let  $e'''$  be a segment that is glued to the cylinder with the travel time  $t'''$ .

We can compose an infinite graph from the edges obtained earlier (see Pic. 1).



Pic. 1. Equivalent graph.

**Theorem 2.1** *For decorated graph obtained by attaching a segment to a flat cylinder, the following asymptotic estimate holds, as  $T$  goes to infinity:*

$$\ln N(T) \leq \sqrt{\frac{2}{3b}} \pi T^{\frac{1}{2}} (1 + o(1))$$

*If times  $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$  are linearly independent over  $\mathbb{Q}$  then inequality turns*

into equality. For almost all real  $a$  and  $b$  times  $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$  are linearly independent over  $\mathbb{Q}$ .

**Proof** We list all times at which  $N(T)$  grows by one. It happens when:

1) A packet arrives at point  $A$ . I.e. at time of a kind:

$$T = t'n' + t''n'' + t'''n''' + \sum_{i=0}^k t_i n_i$$

for some  $k$ , and non-negative integer  $n', n'', n_0, \dots, n_k$  satisfy the following conditions:

- a)  $\sum_{i=0}^k n_i + n'''$  is even
- b) if  $\sum_{i=0}^k n_i + n''' = 0$ , then  $n'' = 0$ ,
- c) if  $\sum_{i=0}^k n_i + n''' > 0$ , then  $n'' \geq 0$ .

A set of paths  $\gamma_T$  from  $A$  to  $A$  corresponds to every moment of time  $T$ .

Function  $N(T)$  does not increase at time  $T$  if the following condition is fulfilled: there is a path in  $\gamma_T$ , by which we return to point  $A$  and we arrive by  $e'''$  at final moment of time. Obviously, there is no such path if and only if  $n''' = 0$ .

Thus a new packet is born in point  $A$  at times of a kind  $t'n + \sum_{i=0}^k t_i n_i$ , where  $n_i \geq 0$ ,  $n \geq 0$ ,  $\sum_{i=0}^k n_i$  is even.

2) A packet arrives at point  $B$ . Similarly, a new packet is born in point  $B$  at times of a kind  $t'n + \sum_{i=0}^k t_i n_i$ , where  $n_i \geq 0$ ,  $n \geq 0$ , and  $\sum_{i=0}^k n_i$  is odd.

Thus  $N(T)$  equals the number of times of a kind  $t'n + \sum_{i=0}^k t_i n_i$ ,  $n_i \geq 0$ ,  $n \geq 0$ , and times less than  $T$ . If  $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$  are linearly independent over  $\mathbb{Q}$ , then there exists one-to-one correspondence between such times and sets  $(n, n_0, n_1, \dots)$ . Hence,  $N(T)$  is equal to the number of solutions of inequality  $t'n + \sum_{i=0}^k t_i n_i \leq T$ . Let us use the theorem 1.1:  $\rho(\lambda) = [\frac{1}{b}\sqrt{\lambda^2 - a^2}]$ , thus  $c_0 = \frac{1}{b}$ ,  $\gamma = 0$  and we obtain

$$\ln N(T) = \sqrt{\frac{2}{3b}} \pi T^{\frac{1}{2}} (1 + o(1)).$$

If  $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$  are linearly dependent over  $\mathbb{Q}$ , then different sets  $(n, n_0, \dots)$  correspond to one moment of time and then

$$\ln N(T) \leq \sqrt{\frac{2}{3b}} \pi T^{\frac{1}{2}} (1 + o(1)).$$

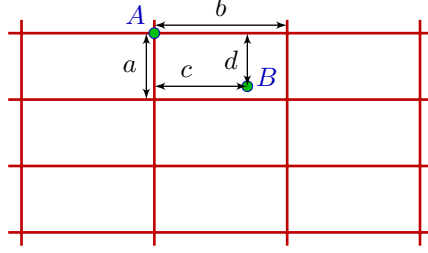
It remains only to explain because of why for almost all real  $a$  and  $b$  times  $\{t'\} \cup \{t_k\}_{k=0}^\infty$  are linearly independent over  $\mathbb{Q}$ . Let us take  $b = 1$  without loss of generality. It is sufficient to prove that a set  $TS = \cup_{k=0}^\infty \{t_k = \sqrt{k^2 + a^2}\}$  is linearly independent over  $\mathbb{Q}$ . Suppose that this is not true. This means that there is a finite list of rational numbers  $\alpha_j, j = 1, \dots, m$  such that finite linear combination of elements of  $TS$  with coefficients  $\alpha_j$  equals zero. Hence the set of  $a$  values such that  $TS$  is  $\mathbb{Q}$ -linearly dependent is smaller than the set of finite sequences of rational numbers. Since the latter set is countable, so is the set of such  $a$ . Hence the set  $TS$  is linearly independent over  $\mathbb{Q}$  for almost all real  $a$ .

**Remark.** One can prove that times  $t_k$  are  $\mathbb{Q}$ -linearly independent if  $a$  is a transcendent number (for example one can take  $a = \pi$ ) in the following manner. Since  $a$  is transcendent, hence  $\mathbb{Q}(a)$  is isomorphic to the field  $\mathbb{Q}(x)$  of rational functions over  $\mathbb{Q}$ . To show that the numbers  $\sqrt{n^2 + a^2}$  are  $\mathbb{Q}$ -linearly independent is therefore equivalent to the statement that the functions  $\sqrt{n^2 + x^2}$  are linearly independent over  $\mathbb{Q}$ . Suppose that there exists a nontrivial linear combination of such functions that equals zero:  $\sum_{j=1}^n \alpha_j \sqrt{j^2 + x^2} = 0$ . We will prove that all  $\alpha_j = 0$  in the following way. We take one summand  $\alpha_j \sqrt{j^2 + x^2}$  and place it on the right hand side. In the neighborhood of the point  $j$  on the complex plane the rest of the sum is holomorphic function, so  $\alpha_j$  should be zero. We can repeat this procedure for all  $j$ .

### 3 Decorated graph obtained by gluing segment to flat torus

Let us take a flat torus with fundamental cycles of lengthes  $a$  and  $b$ . Let us consider a fundamental rectangle with sides  $a, b$  and take points  $A = (0, 0)$ ,  $B = (c, d)$  in it. We glue a segment  $e'''$  with travel time  $t'''$  to points  $A, B$ .





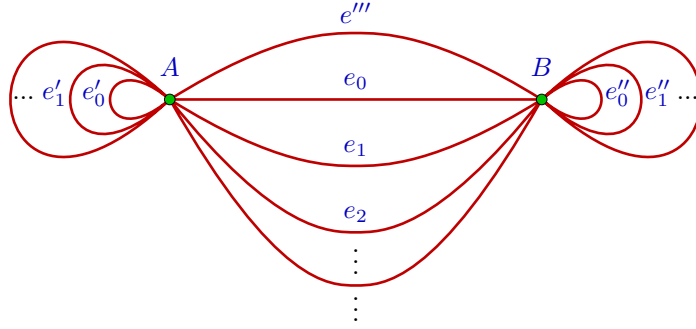
Pic. 2. Gluing of a segment to the flat torus.

Wave packet begins its propagation from point  $A$  and reaches point  $A$  again at times of the following kind  $\{\sqrt{(na)^2 + (mb)^2} | n \geq 0, m \geq 0, n^2 + m^2 \neq 0\}$ . Let us arrange them in ascending order:  $t'_0, t'_1, \dots$ . One of the two front points reaches  $A$  having pass the way  $e'_k$  in time  $t'_k$  (the other point passes this way in the other direction).

In a similar way, we consider front propagating from point  $B$  and we obtain times  $\{t''_i\}_{i=0}^\infty$  (where  $t''_i = t'_i$ ) and paths  $\{e''_i\}_{i=0}^\infty$ .

Front propagating from point  $A$  reaches point  $B$  in times of the following kind  $\{\sqrt{(c+na)^2 + (d+mb)^2} | n, m \in \mathbb{Z}\}$ . Let us arrange them in ascending order:  $t_0, t_1, \dots$ . Let front point reach  $B$  in time  $t_k$  having pass way  $e_k$ .

We can construct an infinite graph from paths obtained before (see Pic. 3).



Pic. 3. Equivalent graph for torus.

**Theorem 3.1** *For a decorated graph obtained by attaching a segment to a flat 2-dimensional torus, the following asymptotic estimate holds, as  $T$  goes to infinity:*

$$\ln N(T) \leq 3 \left( \frac{5\pi}{8ab} \zeta(3) \right)^{\frac{1}{3}} T^{\frac{2}{3}} (1 + o(1))$$

*If  $\{t'_i\}_{i=0}^\infty \cup \{t_i\}_{i=0}^\infty$  are linearly independent over  $\mathbb{Q}$ , then inequality turns into*

equality.

**Proof.** 1) Packets reach point  $A$  at times of the form

$$W = \left\{ t''' n''' + \sum_{i=0} n'_i t'_i + \sum_{i=0} n_i t_i \mid n''' + \sum_{i=0} n_i \text{ is even} \right\}.$$

But it may happen that at such time another packet comes to the point  $A$  by edge  $e'''$ . All such moments are defined by condition  $n''' > 0$ , and  $\sum_{i=0} n_i$  is even. We exclude from  $W$  all such times and obtain

$$W' = \left\{ \sum_{i=0} n'_i t'_i + \sum_{i=0} n_i t_i \mid \sum_{i=0} n_i \text{ is even} \right\}.$$

At each time from  $W'$  a new packet starts from point  $A$  by edge  $e'''$ .

2) Similarly we obtain that at each time from

$$W'' = \left\{ \sum_{i=0} n'_i t'_i + \sum_{i=0} n_i t_i \mid \sum_{i=0} n_i \text{ is odd} \right\}$$

a new packet starts from point  $B$  by edge  $e'''$ .

We join times from cases 1 and 2:  $Q = W' \cup W''$

$$Q = \left\{ \sum_{i=0} n'_i t'_i + \sum_{i=0} n_i t_i \right\}$$

Thus  $N(T) = \{t \in Q \mid t \leq T\}$ .

If  $\{t'_i\}_{i=0}^\infty \cup \{t_i\}_{i=0}^\infty$  are linearly independent over  $\mathbb{Q}$ , then there is a bijection between  $Q$  and sets  $\{n_i\}_{i=0}^\infty \cup \{n'_i\}_{i=0}^\infty$ . Thus  $N(T)$  is equal to the number of inequality solutions  $\sum_{i=0} n'_i t'_i + \sum_{i=0} n_i t_i \leq T$ .

Counting function in our situation is given by the following formula

$$\rho(\lambda) = \#\{\sqrt{(c+na)^2 + (d+mb)^2} \leq \lambda \mid n, m \in \mathbb{Z}\} +$$

$$\#\{\sqrt{(na)^2 + (mb)^2} \leq \lambda \mid n, m \geq 0, n, m \in \mathbb{Z}\} = \frac{5\pi\lambda^2}{4ab}(1 + O(\lambda^{-\varepsilon}))$$

We apply 1.1 with  $c_0 = \frac{5\pi}{4ab}$ ,  $\gamma = 1$  and finish the proof.

If  $\{t'_i\}_{i=0}^\infty \cup \{t_i\}_{i=0}^\infty$  are linearly dependent over  $\mathbb{Q}$  then different sets of integers  $\{n_i\}_{i=0}^\infty \cup \{n'_i\}_{i=0}^\infty$  can correspond to one time of packets birth. It means that Theorem 1.1 gives an upper bound for  $\ln N(T)$ .

Let us consider a decorated graph obtained by gluing an edge  $e'''$  to a 3-dimensional flat torus with fundamental cycle lengths equal  $a, b, c$ . Suppose that edge is glued at points  $A = (0, 0, 0), B = (d, e, f)$ . In this case

$$\begin{aligned}\{t_i\}_{i=0}^\infty &= \{\sqrt{(d+na)^2 + (e+bm)^2 + (f+cl)^2} | n, m, l \in \mathbb{Z}\} \\ \{t'_i\}_{i=0}^\infty &= \{\sqrt{(na)^2 + (bm)^2 + (cl)^2} | n, m, l \geq 0, n, m, l \in \mathbb{Z}\}.\end{aligned}$$

Similarly to the previous theorem we can obtain

**Theorem 3.2** *For a decorated graph obtained by attaching a segment to a flat 3-dimensional torus, the following asymptotic estimate holds, as  $T$  goes to infinity:*

$$\ln N(T) \leq 4 \left( \frac{\pi}{3abc} \zeta(4) \right)^{\frac{1}{4}} T^{\frac{3}{4}} (1 + o(1))$$

*If  $\{t'_i\}_{i=0}^\infty \cup \{t_i\}_{i=0}^\infty$  are linearly independent over  $\mathbb{Q}$  then inequality turns into equality.*

## 4 Uniformly secure manifolds

**Definition.** A Riemannian manifold is said to be *uniformly secure* if there is a finite number  $s$  such that all geodesics connecting an arbitrary pair of points in the manifold can be blocked by  $s$  point obstacles.

Let us remind a theorem that describes uniformly secure manifolds.

**Theorem 4.1** (*K. Burns, E. Gutkin*). *Let  $M$  be a compact Riemannian manifold that is uniformly secure. Then the topological entropy of the geodesic flow for  $M$  is zero, and the fundamental group of  $M$  is virtually nilpotent. If, in addition,  $M$  has no conjugate points, then  $M$  is flat.*

We prove a theorem for decorated graphs constructed from uniformly secure manifolds.

**Theorem 4.2** *Let a segment with the travel time  $L$  be glued at two points  $A$  and  $B$  on the surface  $M$ . Suppose that for geodesics connecting  $A$  with  $A$ ,  $B$  with  $B$ ,  $A$  with  $B$  the following condition holds: the number  $g(\lambda)$  of geodesics whose length is equal or less than  $\lambda$  equals*

$$g(\lambda) = c_0 \lambda^{1+\gamma} (1 + O(\lambda^{-\varepsilon})), \varepsilon > 0.$$

Then

$$\ln N(T) \leq (\gamma + 2) \left( \frac{3c_0 \Gamma(\gamma + 2) \zeta(\gamma + 2)}{(\gamma + 1)^{\gamma+1}} \right)^{\frac{1}{\gamma+2}} T^{\frac{\gamma+1}{\gamma+2}} (1 + o(1)).$$

**Proof** We denote by  $t'_i (i \geq 0)$  the length (i.e. propagation time) of geodesics joining  $A$  and  $A$ , by  $t_i (i \geq 0)$  the length of the geodesic connecting  $A$  and  $B$ , by  $t''_i (i \geq 0)$  the length of the geodesic connecting  $B$  and  $B$ .

Let the packet came from point  $A$ .  $N(T)$  can be increased only in the time of the form (each of these times corresponds to a path from  $A$  to  $A$  or  $A$  to  $B$ )

$$T = \left\{ \sum t'_i n'_i + \sum t''_i n''_i + \sum t_i n_i + L n''' \right\}$$

where  $n''', n_i, n'_i, n''_i$  are non-negative integers such that if  $\forall i : n_i = 0$  and  $n''' = 0$ , then  $\forall i : n''_i = 0$ . Let  $T_A$  be the times in which packets arrive at the point  $A$ . Elements of  $T_A$  are characterized by the condition:  $n'' + \sum n_i$  is even. Similarly, let  $T_B$  be times in which packets arrive at the point  $B$ . Elements of  $T_B$  characterized by the condition:  $n'' + \sum n_i$  is odd.

Then  $T = T_A \cup T_B$ . Now we decompose  $T_A = T_{A_1} \cup T_{A_2}$ , where  $T_{A_1}$  are times satisfying  $n''' \neq 0$ , and  $T_{A_2}$  are times satisfying condition  $n''' = 0$ . Similarly, we decompose  $T_B = T_{B_1} \cup T_{B_2}$ , where  $T_{B_1}$  are times satisfying  $n''' \neq 0$ , and  $T_{B_2}$  are times satisfying condition  $n''' = 0$ .

Note that the number of packets on the edge  $e'''$  will not increase in the time belongs to  $T_{A_1}$  or  $T_{B_1}$  because always there exists packets that come from the edge  $e'''$  at this time.

Therefore,  $N(T)$  does not exceed the number of solutions of the inequality  $\sum t'_i n'_i + \sum t''_i n''_i + \sum t_i n_i \leq T$ . Analogically,  $N(T)$  does not exceed the number of solutions of the inequality  $\sum s_i n_i \leq T$ .

**Consequence 4.1** *Let a segment with the travel time  $L$  be glued at two points  $A$  and  $B$  on the surface  $M$ . Suppose that  $M$  is uniformly secure. Then*

$$\ln N(T) \leq (q + 2) \left( \frac{3C_g \Gamma(q + 2) \zeta(q + 2)}{(q + 1)^{q+1}} \right)^{\frac{1}{q+2}} T^{\frac{q+1}{q+2}} (1 + o(1)),$$

where  $C_g$  and  $q$  depends only on  $M$ .

**Proof** We use the following theorem (see [17]):

**Lemma 4.1** (*K. Burns, E. Gutkin*). *Let  $M$  be a compact Riemannian manifold. If  $M$  is uniformly secure, then there are positive constants  $C$  and  $q$  such that for any pair  $x, y \in M$  we have  $CF_{(x,y)}(T) \leq C_g T^q$ .*

So the number of geodesics grows polynomially. Then we apply our previous theorem and get the result.

## 5 Further questions

We have obtained estimates for  $\ln N(T)$  for the situation where the number of geodesics grows polynomially. Such surfaces are not only the cylinder and flat tori. For all manifolds with polynomial growth it is possible to carry out similar calculations and obtain an upper bound for the number of Gaussian packets.

But, as already mentioned above, for a large class of Riemannian manifolds the number  $CF_T(x, y)$  grows as  $e^{hT}$ . In this case Theorem 1.1 is not applicable and the question of the asymptotic behavior of  $N(T)$  remains open. Moreover, the arithmetic properties of geodesic lengths affect the accuracy of the estimate. Therefore, it would be interesting to consider results related to the linear independence over  $\mathbb{Q}$  of the lengths of geodesics joining two given points on a surface.

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